4 Theory of computation

4.1 Abstraction and automation

Learning objectives:

- *Be able to develop solutions to simple logic problems.*
- *Be able to check solutions to simple logic problems.*

Key term

Proposition:
A proposition can be defined as a statement which can be either true or false, e.g. "Today is Friday"

Information

An argument:
An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion.
An argument is valid if the truth of all its premises implies that the conclusion is true.

Argument form:
An argument form in propositional logic is a sequence of propositions involving propositional variables, e.g. P, Q. An argument form is valid if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.
For example, a given set of propositions $P_1, P_2, P_3,....P_n$, called premises, yields another proposition Q called the conclusion. Such an argument is denoted by the argument form

$$P_1, P_2, P_3,....P_n \vdash Q$$

$P_1, P_2, P_3,....P_n$ and Q are propositional variables. The symbol $\vdash$ denotes an assertion.

4.1.1 Problem solving

Developing and checking solutions to simple logic problems

It is well-known that it snows only if it is cold. "It snows only if it is cold" is called a **proposition**.

> **A proposition can be defined as a statement which can be either *true* or *false*.**

For example, "Today is Friday" is a proposition which is *true* or *false*. If today is Friday then it is *true*, if it isn't Friday today it is *false*.

The following are alternative phrasings of the proposition "It snows only if it is cold":

- "If *it snows*, *it is cold*"
- "If *it is snowing*, *it is cold*"
- "If *it snows* then *it is cold*"
- "If *it is snowing* then *it is cold*"
- "A necessary condition for *it to snow* is *it must be cold*"
- "*Being cold* is necessary for *it to be snowing*".
- "*It is cold* when *it snows*"
- "*It is cold* whenever *it snows*"
- "*It is cold* if *it snows*"
- "*It is snowing* implies *it is cold*"
- "*It is cold* follows from *it is snowing*"
- "a sufficient condition for it *to be cold* is *it is snowing*".

Question

1 Write each of these statements in the form "*If P, then Q*".
(a) It snows whenever the wind blows from the north.
(b) Getting unfit follows from not exercising enough.
(c) Leaves on the trees turning brown implies it is autumn.
(d) Being at a temperature of 100 °C is necessary for water to boil.
(e) A sufficient condition for a refund is that you bought the goods in the last two weeks.

Compound proposition

Actually, "It snows only if it is cold" is a compound proposition because it is made up of

- two simpler propositions "It snows" and "It is cold" and
- the conditional connective, "only if".

If we know that "It snows only if it is cold" is *true* then "It snows" or "It is snowing" is *true* only if "it is cold" is *true*.

We use the major proposition and the minor proposition as shown below **to infer** or argue **by deduction** that if "*If it is snowing then it is cold*" is *true* and if "*It is snowing*" is *true* we may conclude that "*It is cold*" is *true*.

Major proposition: *If it is snowing then it is cold*

Minor proposition: *It is snowing*

Conclusion: *Therefore, it is cold*

Now consider

Major proposition: *If it is snowing then it is cold*

Minor proposition: *It is cold*

Conclusion: *Therefore, it is snowing*

Is this a valid conclusion?

The answer is no. It can be cold and not snowing at the same time. We are not entitled to deduce that it is snowing just because it is cold.

Question

2 Given that the two propositions are true
- If it is cold, he wears a hat
- It is cold

What conclusion can you draw?
Layout your argument in the form of a major and a minor proposition followed by a conclusion as shown above.

4 A politician says in his manifesto:
"*If I am elected, then I will lower taxes.*"
The politician is elected.
Assuming that the two statements are true, what conclusion are you entitled to draw?

6 A politician says in his manifesto:
"*If I am elected, then I will lower taxes.*"
The following are true statements:
The politician is elected.
The politician doesn't lower taxes.
Is the politician's manifesto statement a true statement?

3 Given that the two propositions are true
- If it is cold, he wears a hat
- He wears a hat

Are you entitled to deduce that "it is cold"?
Explain your answer.

5 A politician says in his manifesto:
"*If I am elected, then I will lower taxes.*"
The politician manages to get taxes lowered.
Assuming that the two statements are true, are you entitled to deduce that the politician got elected? Explain your answer.

7 A student, Alice, says to another student, Ben:
"*If you have a valid network password, then you can log onto the network.*"
Ben replies
"*I have a valid network password.*"
What are the logical conditions that must apply to these two statements for the conclusion "*Ben can log onto the network*" to be true?

Suppose without looking, you take a card from a pack of 52 playing cards and place it face down on the table and say "If the card is the King of Diamonds, then it is red." Anyone familiar with the colour of playing card suits (Diamonds and Hearts are red, Spades and Clubs are black) will know that this statement is true.

If you now turn over this card to reveal that it is the Queen of Spades, a black card, is it still true that "If the card is the King of Diamonds, then it is red"? The answer is, of course, yes.

Table 4.1.1 shows the truth table for the proposition "If the card is the King of Diamonds, then it is red." The only case when this proposition is *false* is when "The card is the King of Diamonds" is true and "It is red" is false.

If the card is not the King of Diamonds then it could be red or black in colour but the column headed "If the card is the King of Diamonds, then it is red." will still be *true* for these rows.

The card is the King of Diamonds	It is red	If the card is the King of Diamonds, then it is red
True	True	True
True	False	False
False	True	True
False	False	True

Table 4.1.1 Truth table for the truth of the proposition "If the card is the King of Diamonds, then it is red"

Major proposition: The card is the King of Diamonds only if it is red

Minor proposition: The card is the King of Diamonds

Conclusion: *Therefore, it is a red card*

Figure 4.1.1 Valid argument and a conclusion which is true

Major proposition: The card is the King of Diamonds only if it is red

Minor proposition: The card is not the King of Diamonds

Conclusion: Therefore, *?*

Figure 4.1.2 Valid argument but the conclusion is inconclusive

In *Figure 4.1.1*, the conclusion "*It is a red card*" is true because both major and minor propositions are true.

In *Figure 4.1.2*, both major and minor propositions are true. However, we cannot draw any conclusion other than the card could be red or black because we know that cards are either red or black. But restricting ourselves to the evidence supplied, we are not entitled to use this additional knowledge.

Question

8 Draw the truth table for the proposition "*If I am elected, then I will lower taxes.*"
State the truth values for the row of your table which corresponds to the politician breaking his promise?

Conditional connective ⇒

If we know that "*It is snowing*" is true only if "*It is cold*" is true then using propositional variables, P and Q, where P = "It is snowing", Q = "It is cold", we write $P \Rightarrow Q$ which reads P is true only if Q is true or P implies Q.
In the conditional statement $P \Rightarrow Q$, P is called the hypothesis (or antecedent or premise) and Q is called the conclusion (or consequence). So if the conditional statement is true and hypothesis is true then the conclusion is true.

The argument

Major proposition: *If it is snowing then it is cold*

Minor proposition: *It is snowing*

Conclusion: *Therefore, it is cold*

is a valid argument with a conclusion which is true.

Information

Conditional connective ⇒:

If we know that "*It is snowing*" is true only if "*It is cold*" is true then using propositional variables, P and Q, where P = "*It is snowing*", Q = "*It is cold*" we write

$$P \Rightarrow Q$$

which reads P is true only if Q is true.

In the conditional statement $P \Rightarrow Q$, P is called the hypothesis (or antecedent or premise) and Q is called the conclusion (or consequence). So if the conditional statement is true and the hypothesis is true then the conclusion is true.

Using propositional variables, the equivalent **argument form** of the above **argument** is where ∴ is the symbol that denotes "therefore".

Whenever, both P and P ⇒ Q are true then Q is also true.

$$\begin{array}{l} P \Rightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

The form of the argument that uses propositional logic variables, e.g. P and Q, is called the **argument form** to differentiate it from the **argument** which uses actual statements such as "It is snowing".

Is the conclusion always true if the argument is valid?

Consider the following valid argument

Major proposition: *If computer program A compiles then it will terminate when run*

Minor proposition: *Computer program A compiles*

Conclusion: *Therefore, it will terminate when run*

Even though this is a valid argument because the argument form is the conclusion is not true if the major proposition is not true.

It is a fact that some programs can get stuck in a loop and therefore will not terminate.

$$\begin{array}{l} P \Rightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

Question

9 Consider the following argument involving propositions:
"*If the card is the King of Diamonds, then it is red.*"
"*The card is the King of Diamonds.*"
Therefore, "The card is red."
Write the argument form using the propositional variables, P and Q, where P = "*The card is the King of Diamonds*" and Q = "*It is red.*" Is this a valid argument? Is the conclusion true?

10 The following argument consisting of two propositions and another proposition, the conclusion, is a valid argument but the conclusion is not true even though it is snowing. Why is the conclusion not true?

If it is snowing, it is warm.
It is snowing.
____
Therefore, it is warm.

11 The following valid argument consists of three propositions, one of which is the conclusion. The first two propositions are true. Is the conclusion true?

If it is snowing then it is cold.
It is not cold.
____
Therefore, it is not snowing.

12 The following valid argument consists of three propositions, one of which is the conclusion. The first two propositions are true. Is the conclusion true?

It is snowing or it is cold or it is both.
It is not snowing.
____
Therefore, it is cold.

Logical reasoning

A fundamental principle of logical reasoning states:

If P then Q, If Q then R, therefore, If P then R

that is, the following argument is valid

$$P \Rightarrow Q,\ Q \Rightarrow R \vdash P \Rightarrow R$$

Consider the following statements

- If it is freezing, the streets are icy.
- If the streets are icy, accidents will happen

Applying the fundamental principle of logical reasoning we get

It is freezing implies the streets are icy	*It is freezing* ⇒ *the streets are icy*	$P \Rightarrow Q$
the streets are icy implies accidents will happen	*the streets are icy* ⇒ *accidents will happen*	$Q \Rightarrow R$
It is freezing implies accidents will happen	*It is freezing* ⇒ *accidents will happen*	$\therefore P \Rightarrow R$

Question

13 *If it snows today, then I will not go to school today. If I do not go to school today, I can catch up on my homework.*
Use logical reasoning to draw a conclusion.

14 *If the train arrives early then Jamin will be early for his meeting. If Jamin is early for his meeting, he can call in on his friend John.* What conclusion can you draw?

15 *If I don't watch the late night film, I will go to bed early. If I go to bed early, I will wake up feeling refreshed.* What conclusion can you draw?

16 Consider the following scenario:

If Alex is allowed a TV in his bedroom, he will neglect his homework. If he neglects his homework, he will fall behind at school. If he falls behind at school, he will not progress at school. If he does not progress at school, he will need extra tution.

Use logical reasoning to draw a conclusion.

17 *If the plane arrives late and there are no taxis at the airport then Jack is late for his meeting. Jack is not late for his meeting. The plane did arrive late.* What conclusion can you draw?

18 *If it is raining and Isla does not have her raincoat with her, then she will get wet. Isla is not wet. It is raining.* What conclusion can you draw.

19 Consider this rule about a set of cards:

"If a card has a vowel on one side, then it has an even number on the other side."

Look at the cards below and answer the question which cards do I need to turn over to tell if the rule is actually true? Explain your reasoning.

U L 8 3

Did you know?

Mathematical induction: Unfortunately, the term "mathematical induction" clashes with the terminology used in reasoning by logic. Deductive reasoning uses rules of inference to draw conclusions from premises, whereas inductive reasoning makes conclusions only supported, but not guaranteed, by evidence. Proofs that use mathematical induction are deductive, not inductive.

Did you know?

Rationalism and empiricism: Rationalism is the philosophy that believes that the senses deceive and that logical reasoning is the only sure path to knowledge.
Empiricism is the philosophy that believes that all reasoning is fallible and that knowledge must come from observation and experimentation.
Empiricism gave birth to the scientific method during the Enlightenment.
Rationalism was responsible for the invention of some fundamental mathematical techniques such as calculus but on theories of the universe did not do so well. The empiricists' approach of generalising from what has been observed and measured has been more successful. This approach has also proved useful in machine learning.

From deduction to induction

Newton is well known for his three laws of motion. He is less well known for his four rules of induction. His third rule of induction is said to be at the heart of the Newtonian revolution and modern science. This rule, known as Newton's Principle, can be expressed in modern parlance as follows

> *Newton's Principle: Whatever is true of everything before our eyes is true of everything in the universe.*

Newton's principle allows scientists to draw conclusions of a general nature from the results of experiments involving a limited set of observations and measurements.

We have learned so far about deductive reasoning, e.g.

John Smith is a footballer

All footballers kick a football when playing soccer

Therefore, ...?............

The first statement is a fact about John Smith and the second is a general rule about footballers. We can infer from these two statements that *John Smith kicks a football when playing soccer* by applying the general rule to John.

Inductive reasoning is the inverse of **deductive reasoning**. In inductive reasoning we start instead with the initial and the derived facts, and look for a rule that allows the derived fact(s) to be inferred from the initial fact(s). For example,

John Smith is a footballer

..........?............

Therefore, John Smith kicks a football when playing soccer

One such rule is: *If John Smith is a footballer, then he wears football boots when playing soccer*

This is a rule but not a very useful one because it is specific to John Smith. We use Newton's principle to generalise this rule to make a more useful general rule:

If x is a footballer, then x wears football boots when playing soccer

where x can be anyone including people who are not footballers.

Another way of expressing this is

All footballers wear football boots when playing soccer

We would need more evidence to have faith in this general rule. This could be obtained by observing more than just one footballer playing soccer and noting over and over again that they wear football boots when playing soccer.

Note that the statement

If x is a footballer, then x wears football boots when playing soccer

is neither true nor false whilst the value of x is unknown, but if x = *John Smith* then it becomes

If John Smith is a footballer, then John Smith wears football boots when playing soccer

This is either true or false depending on whether John is a footballer. It is therefore a proposition.

In this logic we have statements that are not propositions until a value has been assigned to the variable(s) in the propositional function.

If we denote

If x is a footballer, then x wears football boots when playing soccer by P(x)

then P("John Smith") is a proposition. P(x) is said to be the value of the propositional function P at x.

Question

20 Let P(x) denote the statement "x > 5". What are the truth values of P(3) and P(7)?

21 The following statements are true:

Gerry is a computer science student.

All computer science students drink coffee.

What conclusion can you draw?

22 The following statements are true:

Deemei is in class CS1.

Every student in class CS1 *has learned at least one programming language.*

What conclusion can you draw?

Extension Question

23 A boy and a girl, whose names we do not know, are sitting next to each other. One of them has fair hair, and the other dark hair. The dark-haired one says "*I am a girl.*" The fair-haired one says, "*I am a boy*". We are told at least one of them is lying. What is your conclusion?

In this chapter you have covered:

- Developing solutions to simple logic problems.
- Checking solutions to simple logic problems.